

# KINEMATICS – the description of motion

## The Big Picture

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Kinematics is about understanding how to describe motion using precise concepts, graphical methods and mathematical equations. You will apply these methods to the solution of problems involving motion in a straight line, such as the motion of falling bodies. You will also understand how non-linear types of motion arise, and will study one such type (projectile motion) in some detail.

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## Learning Objectives

Studying these notes and working through the associated exercises and activities in the Kinematics epack will help you to achieve the following:

- Be able to define displacement, speed, velocity and acceleration
- Recognize and use the vector nature of displacement, velocity and acceleration
- Use graphical representations of motion
- Derive equations of motion for uniform acceleration in a straight line
- Solve problems on linear motion with constant acceleration
- Describe qualitatively the effect of air resistance on motion of falling bodies
- Recognize conditions for the occurrence of non-linear motion: parabolic, circular, simple harmonic motion
- Describe and explain projectile motion (parabolic motion)

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## Introduction to Kinematics

Kinematics is a branch of Mechanics that deals with the *description of motion* in mathematical terms. In kinematics, the *cause of motion* is not considered. This is dealt with in another branch of Mechanics, called Dynamics.

In kinematics we are interested in describing motion using precise concepts and mathematics. Think for example of a moving car. When we talk about the motion of the car, we may be interested in the following:

- How *far* the car moves
- How *fast* it moves
- Whether it *speeds up* or *slows down*

In everyday language we have special words for these things. We talk of:

- **Distance** or **displacement**, for how *far* objects move
- **Speed** or **velocity**, for how *fast* objects move
- **Acceleration** or **deceleration**, for when objects *speed up* or *slow down*

In physics, these same words are given precise meanings – once *defined* precisely, they form the basic concepts of kinematics.

## Definitions

The following quantities will be defined: displacement, velocity, speed and acceleration.

### Displacement

#### DEFINITION

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The **displacement** of a body is specified by the distance it moves together with the direction of its motion.

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Remarks:

- 1) From the above definition, it is evident that displacement is a vector quantity, since it has both magnitude and direction.
- 2) The symbol for displacement is **s** or **x**.

## Velocity

### DEFINITION

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**Velocity** is the rate of change of displacement with time.

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Remarks:

- 1) Velocity is also a vector quantity
- 2) The symbol for velocity is usually **v** or **u**
- 3) The SI unit of velocity is  $\text{ms}^{-1}$

The **average velocity** (or **mean velocity**) of a body is defined as:

*resultant displacement*  
*time interval* , or in symbols, by the *defining equation*

$$\mathbf{v}_{av} = \frac{\Delta \mathbf{s}}{\Delta t} \quad (1)$$

The **instantaneous velocity** (velocity at an instant) is the limiting value of this average velocity when the time interval becomes vanishingly small. In Calculus notation it is given by the defining equation

$$\mathbf{v} = \lim \frac{\Delta \mathbf{s}}{\Delta t} = \frac{d\mathbf{s}}{dt} \quad (2)$$

## Speed

### DEFINITION

**Speed** is the rate of change of distance with time

Remarks:

- 1) Speed is a scalar quantity – it has no direction
- 2) It has the same units and symbols as velocity (but not in bold type)
- 3) It is distance moved per unit time – thus speed (rather than velocity) is the quantity we are more familiar with in everyday life

The **average speed** (or **mean speed**) of a body is defined as:

$$\frac{\text{total distance travelled}}{\text{time interval}}$$

The **instantaneous speed** is the limiting value of this average speed when the time interval becomes vanishingly small.

The defining equations for average and instantaneous speed are similar to those for velocity, with the vector symbols replaced by scalar ones.

### CAUTION

The instantaneous speed is equal to the magnitude of the instantaneous velocity. However, the average speed is not necessarily equal to the magnitude of the average velocity. The interactive activity on 'Velocity' demonstrates this distinction.

### THINGS TO DO NOW

- Can you think of examples when the average speed is or is not equal to the magnitude of the average velocity?

A body moves with constant (or uniform) velocity if both its speed and direction are constant with time. If an object moves at constant speed in a circle, for example, its velocity is not constant because its direction is changing. When its

velocity changes, whether in magnitude or direction, a body is said to accelerate.

## Acceleration

### DEFINITION

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**Acceleration** is the rate of change of velocity with time.

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Remarks:

- 1) Acceleration is a vector quantity
- 2) The symbol for acceleration is usually ***a***
- 3) The SI unit of acceleration is  $\text{ms}^{-2}$
- 4) For uniform velocity acceleration is zero
- 5) For deceleration (decrease in velocity with time), acceleration is negative

**Average acceleration** and **instantaneous acceleration** are defined in the same way as for velocity:

$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t} \quad (3)$$

$$\mathbf{a} = \lim \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \quad (4)$$

General remarks on the definitions:

- These comprise all the quantities you will encounter in Kinematics
- All the above quantities are defined in terms of only two base quantities: length and time. The third base quantity in Mechanics, mass, is not relevant in Kinematics, but becomes so in Dynamics
- All the above definitions must be learnt

**NOTE**

You may wonder where these definitions come from, but it is important to realise that any definition simply states how we want to use a word. A definition is neither true nor false. In kinematics the definitions of speed, acceleration etc are close to how we use these words in everyday usage, but are much more precise, and more importantly, can be expressed in mathematical form.

**THINGS TO DO NOW**

- Go through the interactive activities in the 'Motion Concepts' part of the Kinematics epack.

## Vector Quantities in Kinematics

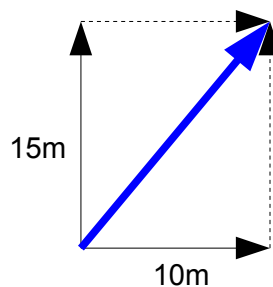
As remarked in the last section, displacement, velocity and acceleration are all vector quantities. This means vector methods must be used for combining them.

### Addition of vectors

To find the sum of two vector quantities such as displacements, we make use of the parallelogram rule for adding vectors.

*Example: A man walks 10m East followed by 15m North. Find his resultant displacement.*

Solution: Here we have two successive displacements which have to be added. Draw a diagram to represent the information:



*Fig 1: Addition of displacements*

The sum of the displacement vectors is given by the parallelogram rule, or equivalently by joining the arrows head to tail. This is shown as the blue arrow.

The magnitude of the displacement is given by (using Pythagoras Theorem)  
 $\sqrt{10^2 + 15^2} m \approx 18.03 m$  and the direction with the horizontal is  $\tan^{-1}(15/10) \approx 56.3^\circ$

### THINGS TO DO NOW

→ Apply a similar method to do the following exercise

*Exercise: Addition of velocities*

*A boat travels at a speed of 5 m/s across a river in a downstream current of 8 m/s. What is the resultant velocity of the boat?*

### Relative velocity

Suppose that two bodies A and B are moving with velocities  $\mathbf{v}_A$  and  $\mathbf{v}_B$  respectively. From the viewpoint of an observer moving with A, B appears to have a relative velocity of  $(\mathbf{v}_B - \mathbf{v}_A)$ , where vector subtraction is implied. Similarly, the velocity of A relative to B is  $(\mathbf{v}_A - \mathbf{v}_B)$ .

The method to obtain vector  $(\mathbf{v}_B - \mathbf{v}_A)$  is shown below.

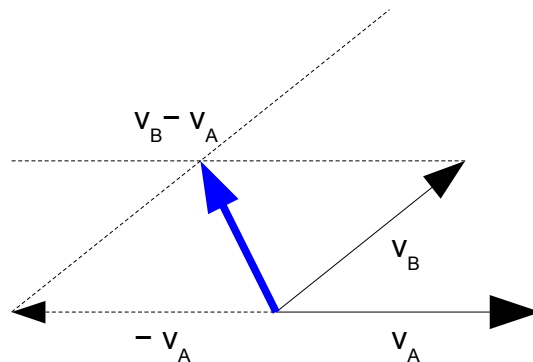


Fig 2: Relative velocity by vector subtraction

**THINGS TO DO NOW**

- Try to think of why the velocity of B from the viewpoint of A is  $(\mathbf{v}_B - \mathbf{v}_A)$ .
- Draw a similar diagram for  $(\mathbf{v}_A - \mathbf{v}_B)$ .

## Graphical Description of Motion

The motion of a body can be represented graphically. There are many different types of graphs. For a particle moving in 2 dimensions, one can plot the y-coordinate of the body against its x-coordinate at different times. The resulting y-x graph would then give the trajectory of the particle in the 2D space. One can also plot the position or velocity components against time.

Two common types of graphs in Kinematics are displacement-time (s-t) and velocity-time (v-t) graphs. We next illustrate these for motion in 1 dimension, where the displacement and velocity have only one component.

### Displacement-time graphs

If we plot the displacement of a body against time we obtain a displacement-time (s-t) graph.

Since  $\mathbf{v} = \lim \frac{\Delta \mathbf{s}}{\Delta t} = \frac{d\mathbf{s}}{dt}$ , this implies that the gradient of a displacement-time graph gives the velocity.

If the velocity is uniform, the displacement-time graph is a straight line. If the velocity varies with time, the displacement-time graph is a curve. In the latter case, the gradient at a point can be obtained by calculating the gradient of the tangent to the curve at that point.

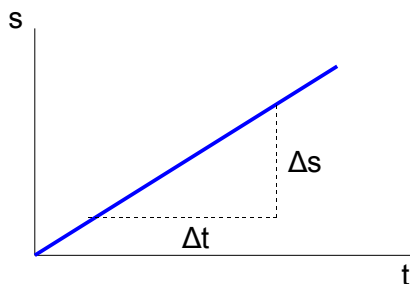


Fig 3a: Uniform velocity

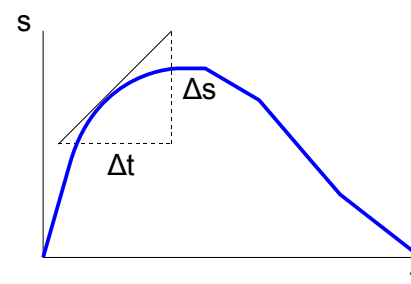


Fig 3b Non-uniform velocity

## Velocity-time graphs

If we plot the velocity of a body against time we obtain a velocity-time (v-t) graph.

Since  $\mathbf{a} = \lim \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$ , this implies that the gradient of a velocity-time graph gives the acceleration.

Figure 4 below shows the velocity-time graph for motion with constant velocity, constant acceleration and variable acceleration respectively.

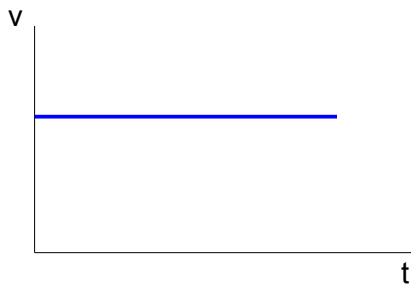


Fig 4a: Constant velocity

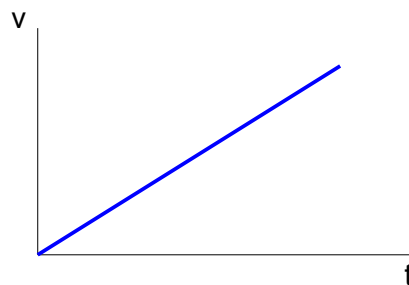


Fig 4b: Constant acceleration

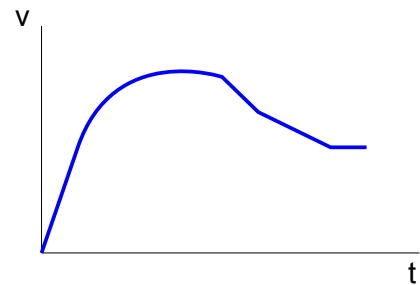


Fig 4c: Variable acceleration

Since  $\mathbf{v} = \frac{d\mathbf{s}}{dt}$ , i.e. velocity is the time derivative of displacement, the displacement between two times  $t_1$  and  $t_2$  can be obtained by integrating the velocity between those two times:

$$\mathbf{s}(t_1) - \mathbf{s}(t_2) = \int_{t_1}^{t_2} \mathbf{v} dt \quad (5)$$

Graphically, the displacement between two times  $t_1$  and  $t_2$  is then given by the area under the velocity-time graph between the times  $t_1$  and  $t_2$ .

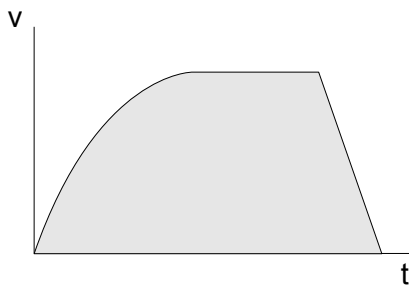


Fig 5: Area under v-t graph gives displacement

**THINGS TO DO NOW**

→ Go through the interactive activities in the 'Graphs' part of the Kinematics epack.

## Equations of Motion under Constant Acceleration

For the special case of motion under constant acceleration (uniform motion) it is possible to derive, starting only from the definitions of velocity and acceleration, a very useful set of equations. These equations of motion can be used to analyze a large number of problems in which the acceleration is constant. This includes the motion of bodies falling under gravity and the motion of projectiles.

We start by deriving the equations of motion, and will then apply them to these problems.

### Derivation of the equations of motion

The starting point for the derivation of the equations of motion is the definition for average acceleration.

$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t}$$

If the acceleration is constant, then at any instant  $\mathbf{a} = \mathbf{a}_{av}$ . This gives

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t}$$

Suppose that initially (i.e. at time  $t = 0$ ),  $\mathbf{v} = \mathbf{u}$  and  $\mathbf{s} = 0$ . Then, at time  $t$

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v} - \mathbf{u}}{t}$$

Rearranging this equation gives

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t \quad (6)$$

This is the first equation of motion. Given the values of the initial velocity and the constant acceleration, it allows the velocity at any other time to be calculated.

It would also be nice to be able to calculate the displacement given the initial velocity and the acceleration. To do this, we use the definition of average velocity to express the displacement  $\mathbf{s}$  after time  $t$  as

$$\mathbf{s} = \mathbf{v}_{av} t$$

Now, the average velocity  $\mathbf{v}_{av}$  is given by

$$\mathbf{v}_{av} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$$

Using Equation (6) to substitute for  $\mathbf{v}$  gives

$$\begin{aligned}\mathbf{v}_{av} &= \frac{1}{2}(\mathbf{u} + \mathbf{u} + \mathbf{a}t) \\ &= \mathbf{u} + \frac{1}{2}\mathbf{a}t\end{aligned}$$

Substituting into the above equation for  $\mathbf{s}$  gives the second equation of motion

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 \quad (7)$$

#### NOTE

Equations (6) and (7) give the velocity  $\mathbf{v}$  and the displacement  $\mathbf{s}$  at any time  $t$ , given the initial velocity  $\mathbf{u}$  and the acceleration  $\mathbf{a}$ , which are both constants.

Therefore, these equations give  $\mathbf{v}$  and  $\mathbf{s}$  as functions of  $t$ , i.e.  $\mathbf{v}(t)$  and  $\mathbf{s}(t)$ .

As noted above, we now have two equations that each respectively give the velocity and displacement as a function of time. In some problems, there may not be information about the time for a motion – it would be useful to also have an equation that relates the velocity and displacement to each other at any time.

To do this, treat Equations (6) and (7) as simultaneous equations and eliminate the time  $t$  from them, to give the third equation of motion

$$v^2 = u^2 + 2as \quad (8)$$

### THINGS TO DO NOW

- Derive Equation (8)
- Reproduce the derivations of Equations (6) and (7) on your own
- Ensure that you can recall all three equations from memory
- If you are familiar with Calculus, derive all three equations by starting from the Calculus definitions of acceleration and velocity, and integrating using the initial condition  $\mathbf{v} = \mathbf{u}$ .

## Application of the Equations of Motion

Collecting the three equations of motion together

$$\begin{aligned} \mathbf{v} &= \mathbf{u} + \mathbf{a}t \\ \mathbf{s} &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 \\ v^2 &= u^2 + 2as \end{aligned} \quad (9)$$

Each of these three equations contains four quantities. Knowing any three of them allows one to calculate the fourth.

### CAUTION

These equations are only valid when the acceleration is constant. If the acceleration changes with time then the equations of motion will be more complicated. To derive the equations for non-uniform acceleration, one needs to go back to the Calculus definitions of acceleration and velocity and integrate with respect to time. You will meet such an example when learning about drag and terminal velocity in Dynamics.

*Example: A car travelling at  $10 \text{ ms}^{-1}$  decelerates uniformly at  $2 \text{ ms}^{-2}$  until it comes to rest. Calculate how long it takes to come to rest. What distance does it travel in that time?*

Solution:

Here we know the initial velocity  $u$  ( $10 \text{ ms}^{-1}$ ), the final velocity  $v$  ( $0 \text{ ms}^{-1}$ ) and the acceleration  $a$  ( $-2 \text{ ms}^{-2}$ ). Hence, we can use the first equation to calculate the time  $t$ .

Using  $v = u + at$  and rearranging gives

$$t = (v - u) / a = (0 - 10) / (-2) \text{ s} = 5 \text{ s}$$

To find the distance travelled in that time, we can use either the 2<sup>nd</sup> or 3<sup>rd</sup> equation.

Using  $v^2 = u^2 + 2as$  and rearranging gives

$$s = (v^2 - u^2) / (2a) = (0 - 10^2) / (2 \times -2) \text{ m} = 25 \text{ m}$$

### THINGS TO DO NOW

→ In the example above, calculate how long the car takes to cover 16 m.

## Motion of Falling Bodies

In this section, we will apply the derived equations of motion (9) to analyse and solve problems on the motion of a body under free fall.

This is a one-dimensional problem, where there is motion only in the vertical direction. Hence, the vector quantities in Equations (9) have only one component, and the equations can be treated as scalar equations.

Earth exerts a gravitational pull on any object. This pull, also known as the weight of the object, is what causes it to fall if released from a height. It has been experimentally observed that bodies under free fall undergo a constant acceleration towards the Earth of about  $9.8 \text{ ms}^{-2}$  (sometimes approximated to  $10 \text{ ms}^{-2}$ ). This is called **acceleration due to gravity** and has the symbol  $g$ . Its value is the same for all falling bodies.

If an object is thrown upwards, it experiences a deceleration of  $9.8 \text{ ms}^{-2}$  (i.e. an acceleration of  $-9.8 \text{ ms}^{-2}$ ) because it is still being pulled down. Therefore, the acceleration vector  $g$  is always directed downwards.

## Time of fall for freely-falling bodies

Consider the application of the Equation (7),  $s = ut + \frac{1}{2}at^2$ , to the motion of a body under free fall.

This is a one-dimensional problem, where there is motion only in the vertical direction. Hence, the vector quantities in Equation (7) have only one component.

Here the acceleration is  $a = g$ .

Suppose the body falls from rest, so that  $u = 0$ .

Applying Equation (7) gives  $s = \frac{1}{2}gt^2$

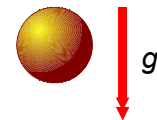


Fig 6: Body under free fall

Rearranging gives

$$t = \sqrt{2s/g} \quad (10)$$

Equation (10) shows that the time for any object to fall a distance  $s$  only depends on  $s$  and  $g$ . It does not depend on the mass or, in fact, any other property of the body.

You have just discovered, like Galileo, that *all bodies fall the same distance in the same time*, independent of their mass.

According to legend, centuries ago Galileo dropped different masses from the top of the leaning tower of Pisa to disprove the common misconception that heavier objects fall faster than lighter ones.

One may wonder, what about very light objects like feathers, or a parachute? They clearly take much longer to fall down. The answer is that, in those cases, air resistance is very significant and retards the fall of the objects. In the derivation of Equation (10), remember that we started out with Equation (7), which was itself derived under the assumption of constant acceleration by neglecting other effects such as air resistance. The effect of air resistance is discussed qualitatively in the next section and quantitatively in the Dynamics epack.

**THINGS TO DO NOW**

- We all know that a feather falls slower than a hammer, because of air resistance. What would you expect to happen if a feather and a hammer were dropped simultaneously in a vacuum?
- David Scott, of the Apollo 15 mission to the Moon, actually performed this experiment on the Moon (where there is, of course, no air). [Click here](#) to watch the video.
- Do the following numerical exercise.

*Exercise: A stone is thrown vertically upwards with an initial velocity of  $20 \text{ ms}^{-1}$ . Calculate the maximum height it reaches and the time it takes to get there. Compare the descent of the stone with its ascent.*

**Effect of air resistance**

When any object moves through air, the air offers a frictional resistance (drag) to the motion. This causes the object to decelerate. The deceleration is not constant but depends on the velocity of the object. The faster the object the greater the resistance and deceleration. You can experience this when you run – the faster you run, the harder the air seems to blow against .

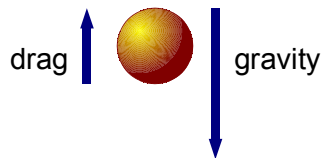


Fig 7: Falling body opposed by drag

If a body falls under gravity, air resistance opposes the fall and the downward acceleration is therefore reduced. This means that bodies falling through air take longer to fall the same distance than in vacuum. The difference is usually slight except for light objects with a large surface area, for which the ratio of air resistance to weight is high. This explains why feathers, parachutes and sheets of paper take much longer to fall than stones.

**THINGS TO DO NOW**

- Go through the interactive activities in the 'Falling Bodies' part of the epack.

## Projectile Motion

In this section, we will apply the equations of motion (9) to analyse and solve problems on the motion of a projectile under a constant acceleration such as gravity.

This is a two-dimensional problem, where there is motion in both the horizontal and the vertical direction. Hence, the vector quantities in Equations (9) have two components each – for each vector equation, there are two equations in terms of the components. The equation for each component must be solved separately.

### THINGS TO DO NOW

→ *Think why each component equation must be solved separately.*

## Non-linear motion

So far we have looked at motion in a straight line (linear motion) under constant acceleration. As we will see in the next section, when two or more dimensions are considered, in general a body subjected to constant acceleration follows a curved path – more specifically a parabolic path.

More complicated types of motion arise if the acceleration is variable. For example, as you will learn later in your study of physics, we can have:

- In one dimension: Simple Harmonic Motion (SHM), where the acceleration is always directed towards a fixed point on a line, and is proportional to the distance from that point.
- In two dimensions: Circular Motion, in which the acceleration is always directed towards the centre of a circle but is of constant magnitude.

## Parabolic motion

Parabolic motion arises when an object moves with constant velocity in one direction whilst being subjected to a constant acceleration in a perpendicular direction.

A common example is that of a projectile in a uniform gravitational field ( another

example is the motion of a charged particle in a constant electric field).

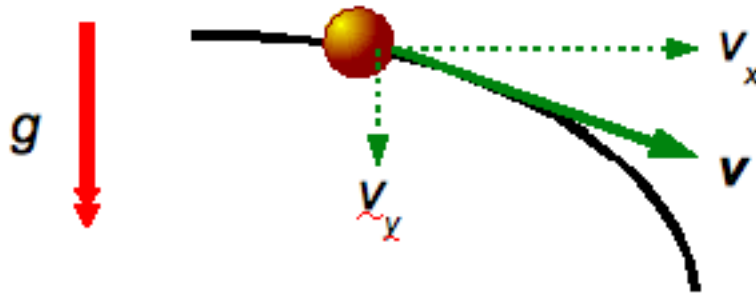


Fig 8: Motion of a projectile

At any point in the projectile's trajectory, the velocity vector  $\mathbf{v}$  can be resolved into its horizontal and vertical components  $v_x$  and  $v_y$  respectively. Now, the acceleration due to gravity  $\mathbf{g}$  only acts vertically and has no component in the horizontal direction. Hence, the horizontal velocity component  $v_x$  remains constant with time, and only the vertical component  $v_y$  is subjected to the downward acceleration  $\mathbf{g}$ .

#### NOTE

Since the horizontal and vertical motions do not affect each other, they can be analysed independently.

#### THINGS TO DO NOW

- Sketch displacement-time, velocity-time and acceleration time graphs for the horizontal and vertical motions of a projectile.

### Time of flight and range of a projectile

Consider a projectile thrown from the ground with speed  $u$  at an initial angle  $\theta$  to the horizontal. We want to find expressions for the **time of flight** (i.e. how long it takes to hit ground again) and for the **range** (maximum horizontal distance reached).

As noted above, the trick is to analyse the horizontal and vertical motions independently.

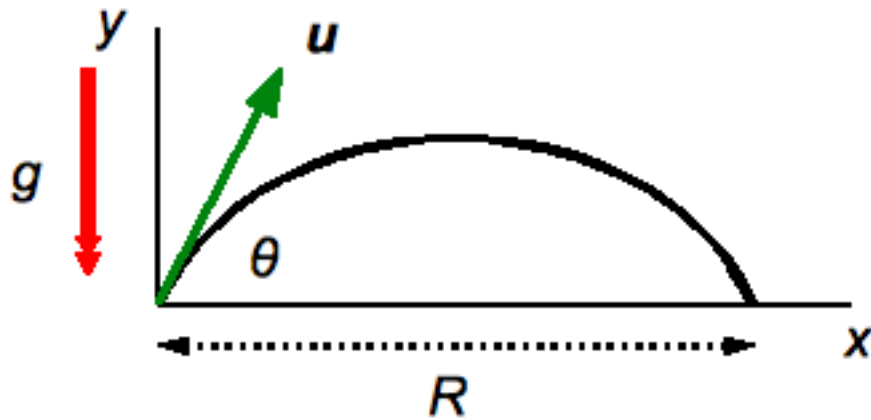


Fig 9: Analysis of projectile motion

Referring to the above diagram,

Horizontal velocity is constant:  $u_x = u \cos \theta$

Initial vertical velocity is:  $u_y = u \sin \theta$  and vertical acceleration is:  $a = -g$

For the vertical motion, using

$$s = ut + \frac{1}{2}at^2$$

and the fact that the projectile hits the ground again when  $s = 0$  (here  $s$  stands for the vertical distance) gives

$$0 = ut \sin \theta - \frac{1}{2}gt^2$$

Rearranging gives the time of flight as

$$t = \frac{2u \sin \theta}{g} \quad (11)$$

The range  $R$  is the horizontal distance travelled in that time, and is therefore given by

$$\begin{aligned} R &= [\text{horizontal velocity}] \times [\text{time of flight}] \\ &= u \cos \theta \times \frac{2u \sin \theta}{g} \end{aligned}$$

Hence, the range is given by

$$R = \frac{u^2 \sin 2\theta}{g} \quad (12)$$

This formula shows that the maximum range is  $u^2/g$  and occurs for  $\sin 2\theta = 1$ , i.e.  $\theta = 45^\circ$ .

#### NOTE

You do not need to remember these formulae, but should be able to derive them from first principles as done above.

#### THINGS TO DO NOW

- Go through the interactive activities in the 'Projectiles' part of the Kinematics epack.
- Do the following exercises.

*Exercise: A ball is thrown horizontally from the top of a cliff with initial velocity  $15 \text{ ms}^{-1}$ . Given that the cliff is 50 m above the ground calculate (i) how long the ball takes to reach the ground, (ii) the horizontal range attained, (iii) the velocity of the ball just before hitting ground.*

*Exercise: Prove that the trajectory of a projectile is a parabola.*